



Regularization of Piecewise Smooth Two-Folds

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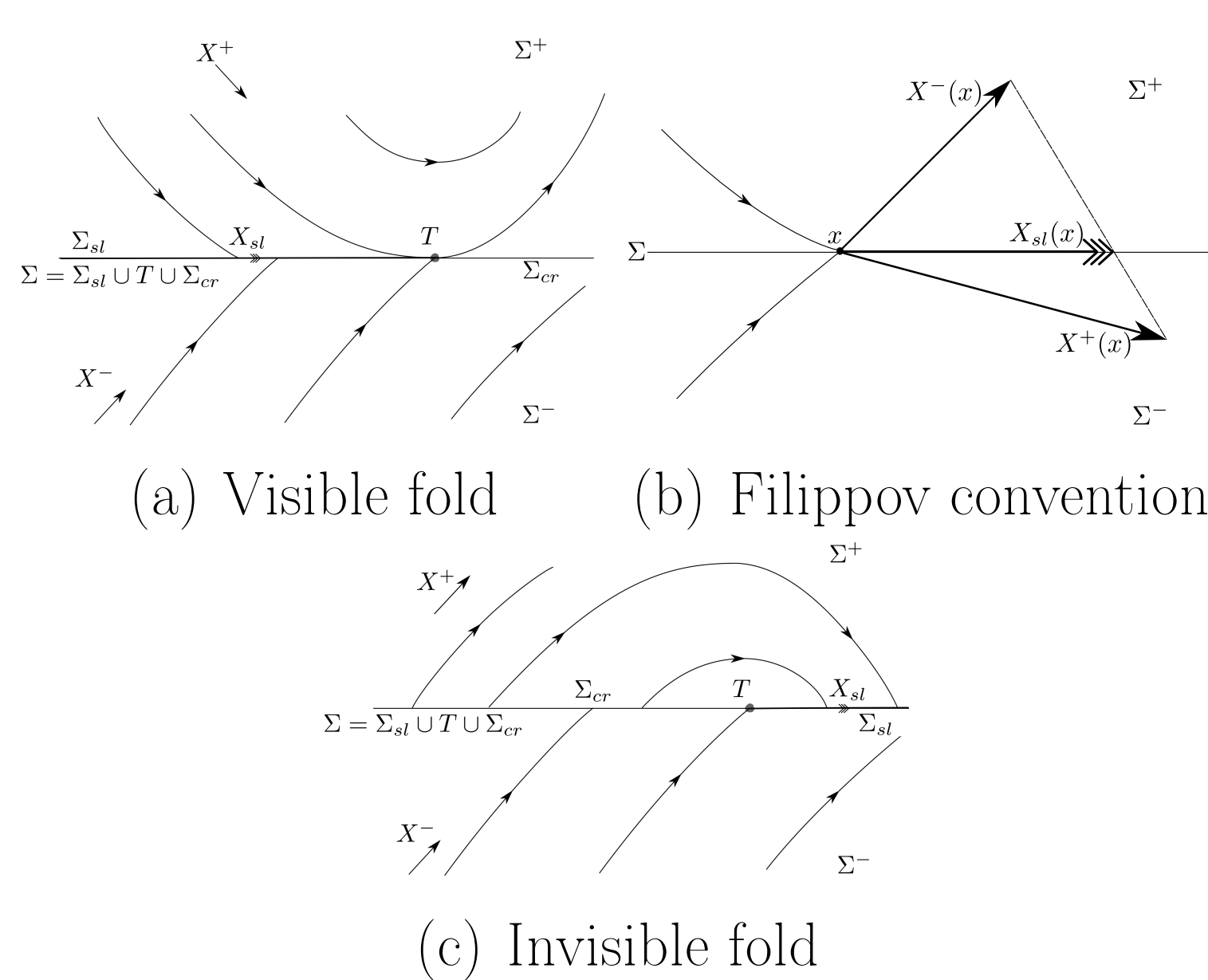
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1 Piecewise Smooth Systems

$X = (X^+, X^-)$ with vector-fields (X^\pm, Σ^\pm) is a piecewise smooth (PWS) system. $\Sigma = \Sigma^+ \cap \Sigma^-$: $f(x, y, z) = 0$ is the switching manifold. Locally we take $f(x, y, z) = y$. Σ is divided into sliding: Σ_{sl} , crossing: Σ_{cr} and tangencies: T , see Fig. (a). On Σ_{sl} we adopt the Filippov convention [2] of sliding (see Fig. (b)) to obtain a vector-field (X_{sl}, Σ_{sl}) .



2 Singularities

$p \in \Sigma$ is a tangency of X^\pm with Σ if $X^\pm f(p) = 0$. A tangency is a **fold** if $X^\pm f(X^\pm f)(p) \neq 0$, being **visible** when > 0 (see Fig. (a)), **invisible** when < 0 (see Fig. (c)). Here $X^\pm f = X^\pm \cdot \nabla f = X_2^\pm$ is the Lie-derivative. A **two-fold** $p \in \Sigma$ is a fold from above and below: $X^+ f(p) = X^- f(p) = 0$.

3 Two-Folds in \mathbb{R}^3

Proposition. [3] Generically, a two-fold p in \mathbb{R}^3 is the transverse intersection of two lines $l^- : x = y = 0, z \in [-c^{-1}, c^{-1}]$, $l^+ : y = z = 0, x \in [-c^{-1}, c^{-1}]$ consisting of fold points of X^\mp , respectively. \square

The lines l^\pm divide $\Sigma : y = 0$ into four separate regions:

- **Stable sliding** $\Sigma_{sl}^- : x \leq 0, z \leq 0$.
- **Unstable sliding** $\Sigma_{sl}^+ : x \geq 0, z \geq 0$.
- **Crossing downwards** $\Sigma_{cr}^- : x \geq 0, z \leq 0$.
- **Crossing upwards** $\Sigma_{cr}^+ : x \leq 0, z \geq 0$.

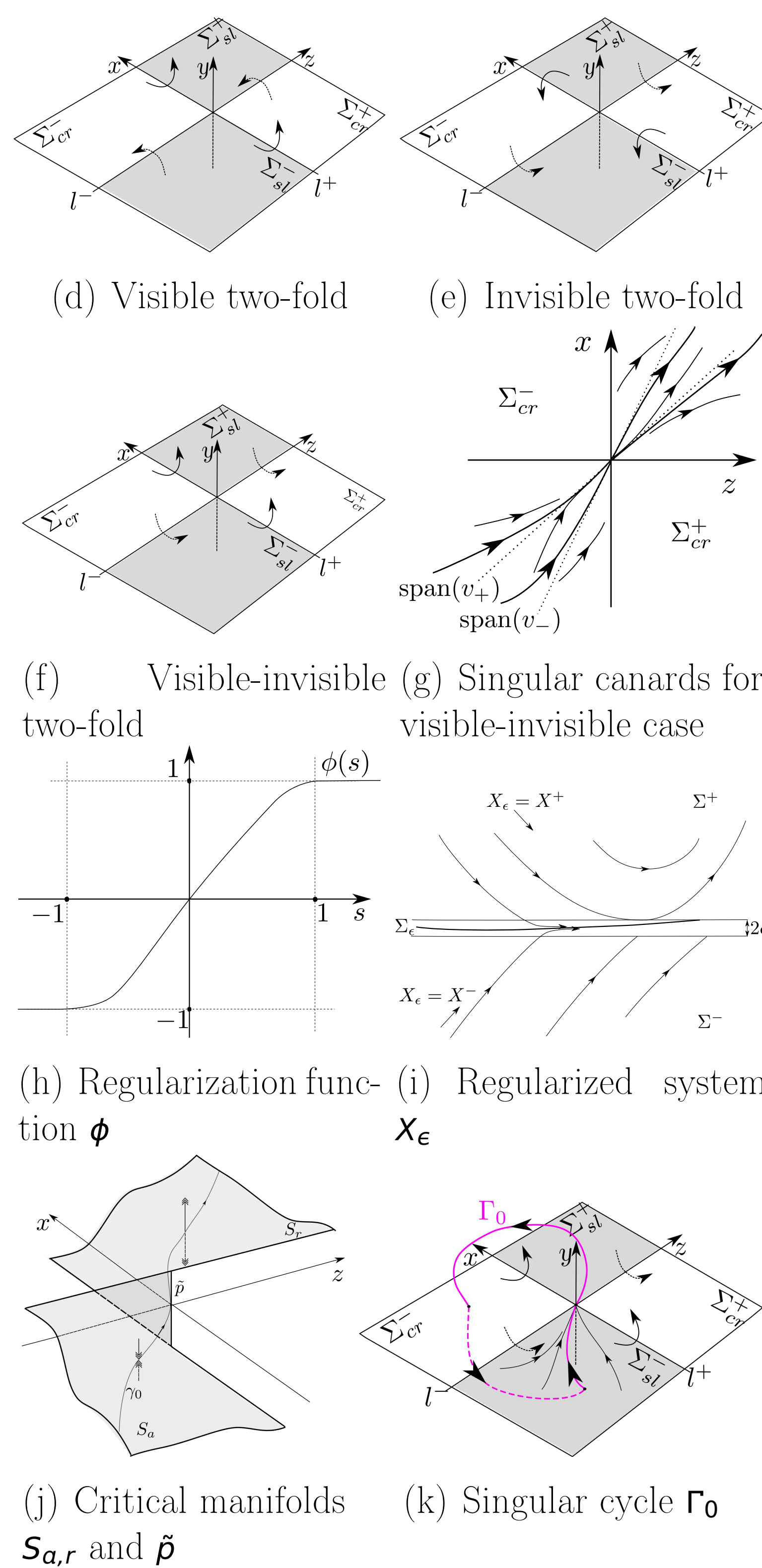
See Fig. (d)-(f). A two-fold is:

- **Visible** if l^\pm are both visible (Fig. (d)).
- **Visible-invisible** if l^+ visible, l^- invisible (Fig. (e)).
- **Invisible** if l^\pm are both invisible (Fig. (f)).

Definition. A **singular canard** of a PWS is a trajectory of $(\Sigma_{sl}^\pm, X_{sl}^\pm)$ having a continuation through the two-fold singularity p . \square

The two-fold p is an equilibrium of the vector-field $F_\mp X_{sl}^\mp$, defined in $\Sigma_{sl}^- \cup \{p\} \cup \Sigma_{sl}^+$ and with $F_\mp = H(x, z)$ a scalar smooth function which is positive (negative) for $x, z < 0$ ($x, z > 0$). Then: **Proposition.** [3] Non-degenerate singular canards exists if and only if p corresponds to a node or a saddle of $F \Sigma_{sl}^-$, and an eigenspace is contained within $\Sigma_{sl} \cup \{p\}$. \square

See Fig. (g).



4 Regularization

- What happens to the two-fold/singular canards when we regularize the PWS system?
- Can we learn something about the PWS system by regularizing?

We consider the Sotomayor-Teixeira regularization [5]:

$$X_\epsilon = \frac{1}{2} X^+ (1 + \phi(\epsilon^{-1} y)) + \frac{1}{2} X^- (1 - \phi(\epsilon^{-1} y)),$$

with $\epsilon \ll 1$ (see Fig. (h) and (i)). Writing $y = \epsilon \hat{y}$ we obtain a hidden slow-fast system with (x, z) slow and \hat{y} fast.

Theorem. [3] X_0 has **critical manifolds**: $S_a = \Sigma_{sl}^-$ (attracting), $S_r = \Sigma_{sl}^+$ (repelling) and a non-hyperbolic line $\tilde{p} : x = z = 0, \hat{y} \in (-1, 1)$ (see Fig. (j)). On $S_{a,r}$: **Reduced system = Filippov sliding system**. \square

Note that in terms of $y = \epsilon \hat{y}$ we have $\tilde{p} = p$.

5 Blowup

To study the persistence of canards we blowup the nonhyperbolic line $\tilde{p} : x = r\bar{x}, z = r\bar{z}, \epsilon = r^2\bar{\epsilon}, (\bar{x}, \bar{z}, \bar{\epsilon}) \in S^2$ following the formulation of Krupa and Szmolyan [4]. We study the phase space using **directional charts** $\kappa_1 : \bar{x} = -1, \kappa_3 : \bar{x} = 1$ and a **rescaling chart**: $\kappa_2 : \bar{\epsilon} = 1$. We obtain:

Theorem. [3] Singular canards \Rightarrow (Primary, maximal) Canards as transverse intersections of continuations of Fenichel slow manifolds $S_{a,\epsilon}$ and $S_{r,\epsilon}$ provided a certain non-resonance condition holds true. These maximal canards are $\mathcal{O}(\sqrt{\epsilon})$ -close to the singular canards. \square

Result and approach very similar to [6, 7] for folds in slow-fast systems in \mathbb{R}^3 . But the geometry is very different.

6 Visible-Invisible Two-Fold

The two-fold is associated with forward and backwards non-uniqueness. By regularizing we can pick the “right orbits”.

Theorem. Consider the visible-invisible case and suppose as in Fig. (k) that there exists a singular cycle Γ_0 (satisfying certain non-degeneracy conditions, see also [1]). Then for $\epsilon \ll 1$ sufficiently small X_ϵ possesses an attracting limit cycle Γ_ϵ satisfying $\Gamma_\epsilon = \Gamma_0 + \mathcal{O}(\sqrt{\epsilon})$. \square

PWS orbit Γ_0 is therefore distinguished, as $\Gamma_0 = \lim_{\epsilon \rightarrow 0} \Gamma_\epsilon$, among all the orbits through p . Note that these results hold true for all monotone regularization functions.

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